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Generalized q-bosons and their squeezed states

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Abstract. Generalized q-boson operators associated with the simultaneous creation of several q-bosons are introduced. These operators give rise to a realization of the quantum Heisenberg–Weyl algebra and are applied to the construction of corresponding Holstein–Primakoff realizations of the quantum algebras $SU_q(2)$ and $SU_q(1,1)$. Coherent states of these algebras are defined in the various ways suggested by the equivalent definitions of the harmonic oscillator coherent states, and some of their properties are studied. Particular attention is devoted to the squeezing properties of the quadratures of the electromagnetic field in these states.

1. Introduction

Deformations of groups and corresponding algebras have been considered for some time (see, for example, [1]). Recent interest in the class of deformations referred to as quantum groups and quantum algebras may have attracted more attention in the physical literature than perhaps is so far warranted by any concrete achievements, since their specific relevance to physical reality has yet to be established. However, we believe that it may prove helpful to possess a reasonable repertoire of techniques and results concerning the properties of these algebras and the wavefunctions they naturally give rise to. One conceivable direction in which this formalism may find worthwhile physical applications is the formulation of approximations to realistic physical situations. Thus, the surprising role of the harmonic oscillator which serves as a paradigmatic approximation to an amazingly rich and diverse set of physical situations, may be considerably enhanced by the availability of a more flexible formalism in terms of a deformed oscillator which contains the familiar oscillator as a special, limiting case. While more ambitious goals, involving a fundamental extension or modification of the structure of quantum mechanics, may be reasonably contemplated with the formalism presently considered, the present article is addressed primarily to the issue of providing the necessary tools for the consideration of the deformed harmonic oscillator in the context of approximating a realistic system within the accepted framework of quantum mechanics.

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2. Generalized q-boson operators

Boson operators satisfy the Heisenberg–Weyl algebra

$$
\begin{aligned}
&[b, b^\dagger] = 1 \\
&[n, b^\dagger] = b^\dagger
\end{aligned}
\tag{1}
$$

where $n = b^\dagger b$.

Generalized boson operators [2] of the form

$$
\begin{aligned}
&B_k = b^k f_k(n) = f_k(n+k) b^k \\
&B_k^\dagger = f_k(n)(b^\dagger)^k
\end{aligned}
\tag{2}
$$

are defined so as to satisfy an algebra isomorphic to that of boson operators, namely

$$
\begin{aligned}
&[B_k, B_k^\dagger] = 1 \\
&[N_k, B_k^\dagger] = B_k^\dagger
\end{aligned}
\tag{3}
$$

where $N_k = B_k^\dagger B_k$. One finds that

$$
f_k(n) = \sqrt{\left[\left|\frac{n}{k}\right|\right] \frac{(n-k)!}{n!}}
\tag{4}
$$

where $[|x|]$ is the integral part of x. Furthermore

$$
N_k = \left[\left|\frac{n}{k}\right|\right] .
\tag{5}
$$

A deformation of the standard boson operator b—the so-called q-boson operator a—satisfying the quantum Heisenberg–Weyl algebra

$$
\begin{aligned}
&a a^\dagger - q a^\dagger a = q^{-n} \\
&[n, a^\dagger] = a^\dagger
\end{aligned}
\tag{6}
$$

with q real and positive, has been recently introduced [3, 4]. The operators a were shown to be realizable in terms of boson operators of the form

$$
a = \sqrt{\frac{[n+1]_q}{n+1}}\, b
\tag{7}
$$

where

$$
[x]_q = (q^x - q^{-x})/(q - q^{-1}) .
\tag{8}
$$

Note that $\lim_{q \to 1} [x]_q = x$, which means that for $q = 1$ the q-boson operators reduce to standard boson operators.

Using equation (7) we find that

$$\begin{aligned} a^\dagger a &= [b^\dagger b]_q = [n]_q \\ a a^\dagger &= [n+1]_q \,. \end{aligned} \tag{9}$$

We now introduce generalized q-boson operators A_k and $A_k^\dagger$ where

$$\begin{aligned} A_k &= a^k f_{k,q}(n) \\ A_k^\dagger &= f_{k,q}(n)(a^\dagger)^k. \end{aligned} \tag{10}$$

The real function $f_{k,q}(n)$ should be determined in such a way that $A_k, A_k^\dagger$ satisfy a relation of the form of equations (6). This is achieved by writing

$$\begin{aligned} A_k &= \sqrt{\frac{[N_k+1]_q}{N_k+1}}\, B_k \\ &= \sqrt{[N_k+1]_q \frac{n!}{(n+k)!}}\, b^k \\ &= \sqrt{[N_k+1]_q \frac{[n]_q!}{[n+k]_q!}}\, a^k \end{aligned} \tag{11}$$

where use has been made of equations (2)–(7). The symbol $[n]_q!$ is defined by

$$[n]_q! = [n]_q [n-1]_q [n-2]_q \cdots [1]_q \,.$$

Note that $B_1 = b$ and $A_1 = a$.

These operators give a multi-boson realization of the quantum Heisenberg–Weyl algebra, thus

$$\begin{aligned} A_k A_k^\dagger - q A_k^\dagger A_k &= q^{-N_k} \\ [N_k, A_k^\dagger] &= A_k^\dagger \,. \end{aligned} \tag{12}$$

They satisfy the properties that one would expect by analogy with the single q-boson case, e.g.

$$\begin{aligned} A_k^\dagger A_k &= [N_k]_q \\ A_k A_k^\dagger &= [N_k+1]_q \end{aligned} \tag{13}$$

and

$$(A_k^\dagger)^i |0\rangle = \sqrt{[i]_q!}\, |ki\rangle \,. \tag{14}$$

3. Holstein–Primakoff realizations of $SU_q(2)$ and $SU_q(1,1)$

The Holstein–Primakoff (boson) realizations of the algebras SU(2) and SU(1,1) are well known [5]. For SU(2) we have

$$\begin{aligned} J_+ &= \sqrt{2\sigma+1-n}\, b^\dagger \\ J_- &= b\sqrt{2\sigma+1-n} = \sqrt{2\sigma-n}\, b \\ J_0 &= n-\sigma \end{aligned} \qquad (15)$$

where σ is the 'angular momentum' quantum number $\sigma = \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

Using equation (1) we find that

$$\begin{aligned} [J_+, J_-] &= 2J_0 \\ [J_0, J_\pm] &= \pm J_\pm \end{aligned} \qquad (16)$$

which are the familiar SU(2) angular momentum commutation relations.

Similarly, the standard Holstein–Primakoff realization of SU(1,1) is given by

$$\begin{aligned} K_+ &= \sqrt{2\sigma-1+n}\, b^\dagger \\ K_- &= b\sqrt{2\sigma-1+n} = \sqrt{2\sigma+n}\, b \\ K_0 &= n+\sigma \end{aligned} \qquad (17)$$

where σ is real and positive. (We shall take 2σ to be integral and positive here.)

These operators satisfy the commutation relations of SU(1,1)

$$\begin{aligned} [K_+, K_-] &= -2K_0 \\ [K_0, K_\pm] &= \pm K_\pm \,. \end{aligned} \qquad (18)$$

A generalization of these Holstein–Primakoff realizations, based on the use of the generalized boson operators of equations (2), was presented in [6]. A Holstein–Primakoff-like realization of the quantum algebra $SU_q(1,1)$ in terms of the q-boson operators was suggested by Chaichian *et al* [7]. They write

$$\begin{aligned} K_+ &= \sqrt{[n]_q}\, a^\dagger \\ K_- &= a\sqrt{[n]_q} \\ K_0 &= n+\tfrac{1}{2} \,. \end{aligned} \qquad (19)$$

These operators satisfy

$$\begin{aligned} [K_+, K_-] &= -[2K_0]_q \\ [K_0, K_\pm] &= \pm K_\pm \,. \end{aligned} \qquad (20)$$

A generalization to arbitrary σ is given by

$$\begin{aligned} K_+ &= \sqrt{[2\sigma - 1 + n]_q}\, a^\dagger \\ K_- &= a\sqrt{[2\sigma - 1 + n]_q} \\ K_0 &= n + \sigma \end{aligned} \tag{21}$$

which also satisfy the $SU_q(1,1)$ relations equations (20) and reduce in the limit $q \to 1$ to realization of $SU(1,1)$ given in equations (17). The corresponding Holstein–Primakoff realization of $SU_q(2)$ can be similarly formulated as

$$\begin{aligned} J_+ &= \sqrt{[2\sigma + 1 - n]_q}\, a^\dagger \\ J_- &= a\sqrt{[2\sigma + 1 - n]_q} \\ J_0 &= n - \sigma \end{aligned} \tag{22}$$

for which

$$\begin{aligned} [J_+, J_-] &= [2J_0]_q \\ [J_0, J_\pm] &= \pm J_\pm\,. \end{aligned} \tag{23}$$

These realizations of $SU_q(2)$ and $SU_q(1,1)$ can be expressed in terms of the conventional boson operators by using equation (7). They can be generalized to realizations in terms of k, q-bosons by replacing a by A_k and n by $N_k \equiv [|\frac{n}{k}|]$. Thus, for $SU_q(2)$ we write

$$\begin{aligned} J_+^{(k)} &= \sqrt{[2\sigma + 1 - N_k]_q}\, A_k^\dagger \\ J_-^{(k)} &= A_k\sqrt{[2\sigma + 1 - N_k]_q} \\ J_0^{(k)} &= N_k - \sigma \end{aligned} \tag{24}$$

in terms of the operators $A_k, A_k^\dagger$ of equations (10). The analogous operators in the multi-photon $SU_q(1,1)$ case are given by

$$\begin{aligned} K_+^{(k)} &= \sqrt{[2\sigma - 1 + N_k]_q}\, A_k^\dagger \\ K_-^{(k)} &= A_k\sqrt{[2\sigma - 1 + N_k]_q} \\ K_0^{(k)} &= N_k + \sigma\,. \end{aligned} \tag{25}$$

The states that we shall consider in this paper are coherent states generated by the *multi-photon* realizations of equations (24) and (25) of $SU_q(2)$ and $SU_q(1,1)$, respectively, in addition to the coherent states generated by the multi-photon realization of the quantum Heisenberg–Weyl algebra, equation (12). In fact, only the cases $k = 1$ or 2, corresponding to the single-photon and two-photon cases respectively, are of interest here as the other cases do not result in squeezing due to the vanishing of most of the expectations (see equations (54)).

Note that the $k = 2$ Holstein–Primakoff realization of SU(1, 1) is not the same (for any σ) as its standard two-boson realization; nor is it in the quantum case for which a realization is given by Kulish and Damaskinsky [8] as

$$\begin{aligned} K_+ &= k(a^\dagger)^2 \\ K_- &= k(a)^2 \\ K_0 &= \tfrac{1}{2}(n + \tfrac{1}{2}) \end{aligned} \tag{26}$$

with $k = (q + q^{-1})^{-1}$ and $[K_+, K_-] = -[2K_0]_{q^2}$. Squeezed states corresponding to this latter case have already been treated by the present authors [9].

4. Generalized coherent states

The familiar (Glauber) coherent states may be defined as eigenstates of b

$$b|\beta\rangle = \beta|\beta\rangle\,. \tag{27}$$

They satisfy

$$|\beta\rangle = \mathcal{N}^{-1}\exp(\beta b^\dagger)|0\rangle \tag{28}$$

where

$$\mathcal{N}^2 = \exp(|\beta|^2)\,. \tag{29}$$

In principle, either equation (27) or equation (28) can be used as a starting point for a definition of q-coherent states. It is easily shown that an attempt to use equation (28) does not lead to a normalizable state. Starting from equation (27), q-coherent states for the q-boson operator a were constructed in [4]; this leads to the (normalizable) state

$$|\beta\rangle_q = \mathcal{N}^{-1}\mathrm{exp}_q(\beta a^\dagger)|0\rangle \tag{30}$$

where

$$\mathcal{N}^2 = \mathrm{exp}_q(|\beta|^2) \tag{31}$$

and

$$\mathrm{exp}_q(x) = \sum_{i=0}^{\infty}\frac{x^i}{[i]_q!}\,. \tag{32}$$

Clearly $\mathrm{exp}_q(x)$ is convergent for all x and $\lim_{q\to 1}\mathrm{exp}_q(x) = \exp(x)$.

The isomorphism between the generalized q, k-boson operator A_k and the q-boson a enables a coherent state for A_k to be written down immediately as

$$|\beta; k\rangle = \mathcal{N}^{-1}\mathrm{exp}_q(\beta A_k^\dagger)|0\rangle \tag{33}$$

with $\mathcal{N}^2$ given by equation (31). In terms of the number eigenstates of the conventional boson operators we have

$$|\beta; k\rangle_q = \mathcal{N}^{-1}\sum_{i=0}^{\infty} c_i|ki\rangle \tag{34}$$

where the coefficients c_i are given by

$$c_i = \frac{\beta^i}{\sqrt{[i]_q!}}. \tag{35}$$

Coherent states for SU(2) and SU(1, 1), as well as the usual Heisenberg–Weyl coherent states, have been considered by several authors (see, for example, [6], [10] and references therein.) The Holstein–Primakoff realization has been used to construct coherent states of the SU(2) and SU(1, 1) groups both in terms of (conventional) boson operators b and (conventional) multi-boson operators B_k [2]. We may use the q-analogues of the Holstein–Primakoff realization given above for $SU_q(2)$ and $SU_q(1,1)$ by equations (24) and (25) to construct coherent states for these latter groups. For $SU_q(2)$ we propose the following three distinct forms:

$$|\beta; k\rangle_{\exp} = \mathcal{N}^{-1}\exp(\beta J_+^{(k)})|0\rangle \tag{36}$$

$$|\beta; k\rangle_{\exp_q} = \mathcal{N}^{-1}\exp_q(\beta J_+^{(k)})|0\rangle \tag{37}$$

and

$|\beta; k\rangle_{\text{eigen}}$ a normalized eigenstate of $J_-^{(k)}$.

(Note that the normalization factor differs in value for each case; the vacuum state is the normalized vacuum for both conventional bosons b and q-bosons a).

These states can be expressed in terms of number eigenstates of the conventional boson operators of the form

$$|\beta; k\rangle_q = \mathcal{N}^{-1}\sum_{i=0}^{2\sigma} c_i |ki\rangle \tag{38}$$

where

$$\mathcal{N}^2 = \sum_{i=0}^{2\sigma} {c_i}^2 \tag{39}$$

and

$$c_{i,\exp} = \beta^i \frac{[i]_q!}{i!} \begin{bmatrix} 2\sigma \\ i \end{bmatrix}_q^{1/2} \tag{40}$$

$$c_{i,\exp_q} = \beta^i \begin{bmatrix} 2\sigma \\ i \end{bmatrix}_q^{1/2} \tag{41}$$

$$c_{i,\text{eigen}} = \frac{\beta^i}{i!} \begin{bmatrix} 2\sigma \\ i \end{bmatrix}_q^{-1/2}. \tag{42}$$

The q-binomial coefficient used in these expressions is given by

$$\begin{bmatrix} r \\ s \end{bmatrix} = \frac{[r]_q!}{[s]_q![r-s]_q!}. \tag{43}$$

For $SU_q(1,1)$ we find that $\exp(\beta K_+^{(k)})|0\rangle$ is not normalizable, so we remain with two possibilities, namely

$$|\beta;k\rangle_{\exp_q} = \mathcal{N}^{-1}\exp_q(\beta K_+^{(k)})|0\rangle \tag{44}$$

and

$$|\beta\rangle_{\mathrm{eigen}} \qquad \text{an eigenstate of } K_-^{(k)}\,.$$

Note that the former state (44) is *not* normalizable for the usual two-boson case equations (26) (see [9]). In the boson number basis we obtain expressions of the form specified in equation (34), with

$$c_{i,\exp_q} = \beta^i \begin{bmatrix} 2\sigma - 1 + i \\ i \end{bmatrix}_q^{1/2} \tag{45}$$

and

$$c_{i,\mathrm{eigen}} = \frac{\beta^i}{i!} \begin{bmatrix} 2\sigma - 1 + i \\ i \end{bmatrix}_q^{-1/2}. \tag{46}$$

The former state is normalizable for $\beta^2 q^{2\sigma-1} < 1$, while the latter is always normalizable.

5. Squeezing properties of generalized coherent states

The two orthogonal components of the (conventional) electromagnetic field are denoted by $x = (b + b^\dagger)/\sqrt{2}$ and $p = (b - b^\dagger)/\mathrm{i}\sqrt{2}$ alluding to the analogous degrees of freedom of the corresponding harmonic oscillator. The uncertainties of these components are given by

$$\begin{aligned}[\Delta x]^2 &= \langle x^2 \rangle - \langle x \rangle^2 = u + v \\ [\Delta p]^2 &= \langle p^2 \rangle - \langle p \rangle^2 = u - v\end{aligned} \tag{47}$$

where, for states specified by real wavefunctions,

$$\begin{aligned}u &= \langle n + 1/2 \rangle - \langle b \rangle^2 \\ v &= \langle b^2 \rangle - \langle b \rangle^2\,.\end{aligned} \tag{48}$$

For the vacuum, and coherent states satisfying equation (27), $v = 0$ and $u = 1/2$; i.e. $\Delta x = \Delta p = 1/\sqrt{2}$ and $\Delta x.\Delta p$ is minimal. States for which either Δx or Δp is less than $1/\sqrt{2}$ are referred to as *squeezed* states ([11, 12]). We may introduce the x_q and p_q components of the q-electromagnetic field in complete analogy to the above by

$$\begin{aligned}x_q &= (a + a^\dagger)/\sqrt{2} \\ p_q &= (a - a^\dagger)/\mathrm{i}\sqrt{2}\end{aligned} \tag{49}$$

which have q-uncertainties given by

$$
\begin{aligned}
[\Delta x_q]^2 &= \langle x_q^2 \rangle - \langle x_q \rangle^2 = u_q + v_q \\
[\Delta p_q]^2 &= \langle p_q^2 \rangle - \langle p_q \rangle^2 = u_q - v_q
\end{aligned}
\tag{50}
$$

where, for real states,

$$
\begin{aligned}
u_q &= \tfrac{1}{2}(\langle [n]_q + [n+1]_q \rangle) - \langle a \rangle^2 \\
v_q &= \langle a^2 \rangle - \langle a \rangle^2 .
\end{aligned}
\tag{51}
$$

For the q-vacuum (which is the same as the ordinary vacuum) and the q-coherent states defined in equation (30) we find that $v_q = 0$, i.e the q-uncertainties satisfy $\Delta x_q = \Delta p_q$. For the other generalized coherent states defined in section 4 we find that one of the two q-uncertainties Δx_q or Δp_q is smaller than the other. This q-uncertainty is referred to as q-squeezed.

We now present the actual expressions for the various expectation values appearing in equations (47) and (49).

For a state of the form $\sum_i c_i |i\rangle$ given in terms of the conventional boson number-eigenstates we have

$$
\begin{aligned}
\langle b^\dagger b \rangle &= \sum_i c_i^2 (i) \\
\langle b \rangle &= \sum_i c_i c_{i+1} \sqrt{i+1} \\
\langle b^2 \rangle &= \sum_i c_i c_{i+2} \sqrt{(i+1)(i+2)}.
\end{aligned}
\tag{52}
$$

For states generated in terms of $A_2^\dagger$, which are of the form $\sum_i c_i |2i\rangle$, we have

$$
\begin{aligned}
\langle b^\dagger b \rangle &= \sum_i c_i^2 (2i) \\
\langle b \rangle &= 0 \\
\langle b^2 \rangle &= \sum_i c_i c_{i+1} \sqrt{(2i+1)(2i+2)} .
\end{aligned}
\tag{53}
$$

For states generated by the $A_k^\dagger$ generalized boson operators with $k \geq 3$ we have

$$
\begin{aligned}
\langle b^\dagger b \rangle &= \sum_i c_i^2 (ki) \\
\langle b \rangle &= 0 \\
\langle b^2 \rangle &= 0.
\end{aligned}
\tag{54}
$$

To calculate the q-uncertainties we have to modify the above expressions (52)–(54) by replacing each factor surrounded by parentheses on the right-hand sides by its q-analogue, that is, $(i) \to [i]_q$, etc.

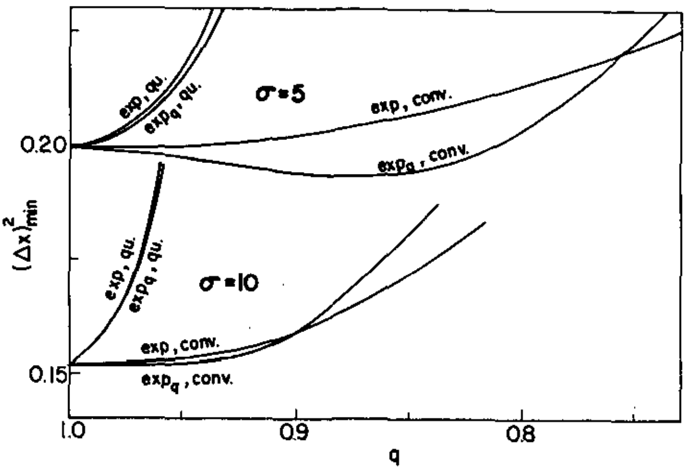

Figure 1. Position squeezing for the $SU_q(2)$ single-photon q-coherent exp and $\exp_q$ states ($\sigma = 5$ and 10).

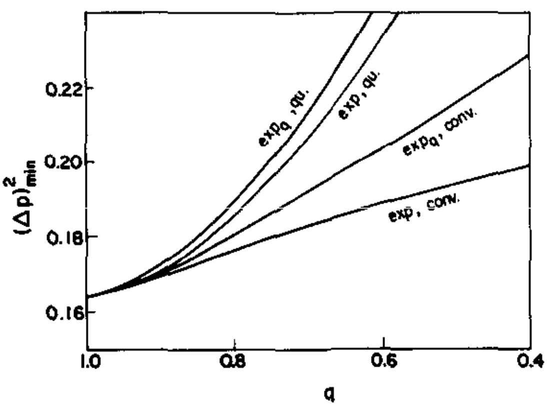

Figure 2. Momentum squeezing for the $SU_q(2)$ two-photon q-coherent exp and $\exp_q$ states ($\sigma = 10$).

6. Computational results

The conventional and quantal squeezing properties of the various coherent states considered above were studied in a series of representative computations, presented as figures 1–4. We consider both a conventional electromagnetic field, i.e. one expressed in terms of the conventional boson operators $b, b^\dagger$ and a 'quantal' field, expressed in terms of the q-bosons $a, a^\dagger$. The study of the conventional uncertainties of the quadratures of the physical electromagnetic field represents the point of view according to which of the various q-coherent states defined in section 4 should be considered as potential approximations to some 'exotic' states of the conventional electromagnetic field. The quantal uncertainties represent a fundamental departure from the conventional theory, which involves the q-observables x_q and p_q. The conventional and quantal uncertainties are evaluated with respect to the various (normalizable) states discussed in section 4, namely:

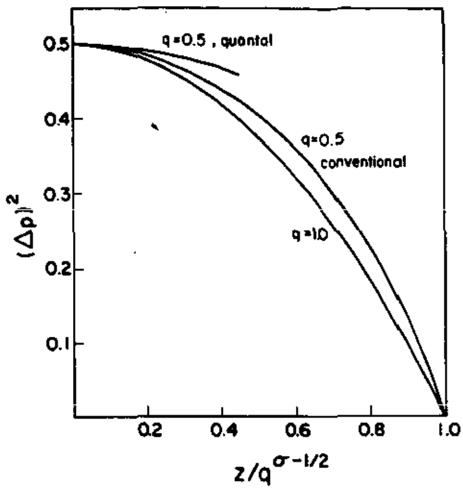

Figure 3. Momentum squeezing for the $SU_q(1,1)$ single-photon q-coherent $\exp_q$ states ($\sigma = 10$).

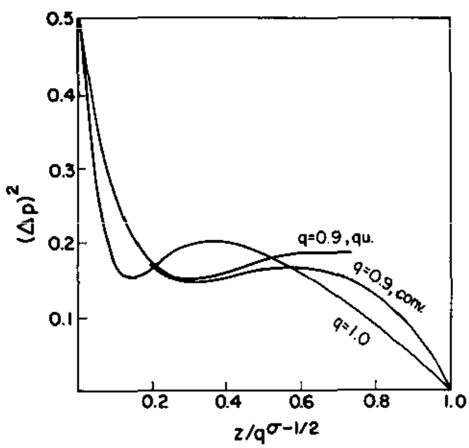

Figure 4. Momentum squeezing for the $SU_q(1,1)$ two-photon q-coherent $\exp_q$ states ($\sigma = 10$).

(i) HW_q coherent state ($\exp_q$ state) (equation (30));

(ii) $SU_q(2)$ states:

(a) exp state (equation (36));

(b) $\exp_q$ state (equation (37));

(c) eigenstate of J_-^k ('step-down eigenstate');

(iii) $SU_q(1,1)$ states:

(a) $\exp_q$ state (equation (44));

(b) eigenstate of K_-^k ('step-down eigenstate').

We now present and discuss some representative results.

As expected, in the $q = 1$ limit the conventional and quantal uncertainties are equal to one another. Furthermore, in the $SU_q(2)$ case, in which both the exp and the $\exp_q$ versions of q-coherent states exist, the distinction disappears in the $q = 1$ limit. This is useful as a check of the computed results. For the single-photon Heisenberg–Weyl q-coherent states, the case $q = 1$ is the familiar coherent state, for which the position and momentum uncertainties are both equal to the minimum uncertainty value of 0.5. In all cases except the HW_q (quantum Heisenberg–Weyl) coherent state expectation of the quantal field there is squeezing; i.e. one or other of the components has a dispersion less than that in the vacuum state. Note that the conventional photon field *is* squeezed in the HW_q coherent state; this case has already been treated by the authors [9]. In some cases we plot the value of the squeezed dispersion against the squeezing parameter z, taken to be real; or against $z/q^{\sigma-1/2}$ to allow for the convergence criterion of the state equation (45), for constant q. Whenever appropriate, we plot the dispersion minimized with respect to the squeezing parameter against the quantum index q, for various values of σ where relevant. As the analysis is symmetric under the interchange $q \to 1/q$ the whole range is covered by abscissal values 0 (extreme quantum algebra) to 1 (standard algebra).

The HW_q two-photon momentum uncertainty agrees in the limit $q = 1$ with the value of 0.15872 determined in [13]. Agreement between the $q = 1$ limit and the corresponding results in [10] is observed for the single- and two-photon uncertainties for the $\mathrm{SU}_q(2)$ q-coherent states, figures 1 and 2. For these states the uncertainties obtain some finite minimal values as functions of the parameter z, for any fixed value of q; cf figures 1 and 2. In some cases the minimal uncertainty vanishes, for all or certain values of q, at the end of the range of z values for which the corresponding state is normalizable, cf figures 3 and 4. Further properties of the conventional and quantal uncertainties computed for the various cases can be read from the figures. The present study is not meant to exhaust the subject. In particular, no effort was made to cover the whole range of the various parameters involved. In addition, no discussion of the time evolution of these phenomena was made, as this would entail postulating a specific model Hamiltonian determining the dynamics.For example,using a reasonable q-analogue of a quantum harmonic oscillator it may be shown that the t-evolution of the position quadrature operator does result in squeezing in the HW quantal case although initially (at $t = 0$) there is no squeezing [14]. The results exhibit a rich variety of phenomena which can only be discussed in rather speculative terms at this stage, a temptation we deliberately avoid. Further thought is needed before a sound assessment of the potential value of these results can be made.

The modified q-boson algebra

$$aa^\dagger - qa^\dagger a = 1 \tag{55}$$

has recently been considered [15]. This algebra can be represented in terms of conventional boson operators as

$$a = \sqrt{\frac{q^{n+1}-1}{(q-1)(n+1)}}\, b\,. \tag{56}$$

The formalism developed in the present paper remains valid provided that $[x]_q$ is redefined to mean $[x]_q = (q^x - 1)/(q-1)$.

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